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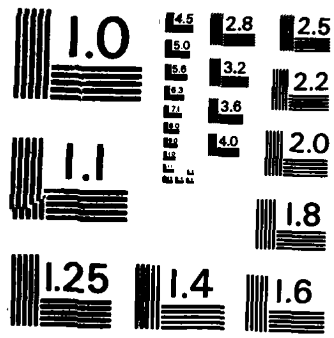

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L^∞ STABILITY FOR WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN $\mathbf{R}^3$

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L^2 STABILITY FOR WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN R^3

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ABSTRACT

In this paper we study L^2 stability for weak solutions of the Navier-Stokes system of equations (1), in the whole space R^3. We assume that it is given a weak solution v satisfying certain hypotheses (essentially $v \in L^{2p/p-3}(0; +\infty; L^p)$, $p > 3$) and we prove that, to an arbitrarily large perturbation of his initial velocity, it corresponds a weak solution u such that the L^2-norm $\|u(t) - v(t)\|$ converges to zero as time t tends to infinity.

AMS (MOS) Subject Classifications: 35B35, 35K55, 35Q10, 76D05

Key Words: nonlinear partial differential equations, viscous fluid motions, initial value problem, asymptotic stability, Navier-Stokes equations

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SIGNIFICANCE AND EXPLANATION

We consider the motion of a viscous fluid filling the whole space $\mathbb{R}^3$, governed by the classical Navier-Stokes equations (1). Existence of global (in time) regular solutions for that system of non-linear partial differential equations is still an open problem. Up to now, the only available global existence theorem (other than for sufficiently small initial data) is that of weak (turbulent) solutions. From both the mathematical and the physical point of view, an interesting property is the stability of such weak solutions. We assume that $v(t,x)$ is a solution, with initial datum $v_0(x)$. We suppose that the initial datum is perturbed and consider one weak solution u corresponding to the new initial velocity. Then we prove that, due to viscosity, the perturbed weak solution u approaches in a suitable norm the unperturbed one, as time goes to $\cancel{+\infty}$, without smallness assumptions on the initial perturbation.

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L^2 STABILITY FOR WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN $\mathbb{R}^3$

Paolo Secchi*

Introduction

Consider the initial value problem for the non-stationary Navier-Stokes equations in the whole space $\mathbb{R}^3$

$$
(1) \qquad
\begin{aligned}
&u' + (u\cdot\nabla)u - \Delta u + \nabla p = f && \text{in } Q_T =]0,T[\times\mathbb{R}^3 ,\\
&\operatorname{div} u = 0 && \text{in } Q_T ,\\
&u|_{t=0} = u_0 && \text{in } \mathbb{R}^3 ,\\
&\lim_{|x|\to\infty} u(t,x) = 0 && \text{for } t \in]0,T[,
\end{aligned}
$$

where $T \in]0,+\infty]$, $u' = \frac{\partial u}{\partial t}$ and $(u\cdot\nabla)u = \sum_{i=1}^{3} u_i \frac{\partial u}{\partial x_i}$. Here u and p denote, respectively, the unknown velocity and pressure; moreover, the given initial velocity u_0 satifies $\operatorname{div} u_0 = 0$ in $\mathbb{R}^3$.

We denote by $L^p = L^p(\mathbb{R}^3)$, $1 \le p \le \infty$, the usual Lebesgue space of $\mathbb{R}^3$-valued functions on $\mathbb{R}^3$ and by $\|\cdot\|_p$ its norm; the L^2-norm is simply denoted by $\|\cdot\|$. Denote by $H^1 = H^1(\mathbb{R}^3)$ the L^2-Sobolev space of order 1 and by H^{-1} the dual space of H^1. Let be the set of all smooth functions φ with compact support in $\mathbb{R}^3$ such that $\operatorname{div} \varphi = 0$. We denote by H the L^2-closure of and by V the H^1-closure of .

By a <u>weak solution</u> of problem (1) we mean a vector $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$, for all $T > 0$, such that

$$\int_0^T \int [u\cdot\varphi' + (u\cdot\nabla)\varphi\cdot u + u\cdot\Delta\varphi + f\cdot\varphi]dxdt = - \int u_0\varphi|_{t=0}dx ,$$

for every regular divergence free vector field $\varphi(t,x)$, with compact support with respect to the space variables and such that $\varphi(T,x) = 0$. The weak solution u is also assumed to satisfy the following energy inequality:

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$$\|u(t)\|^2 + 2\int_0^t \|\nabla u\|^2 ds \le \|u_0\|^2 + 2\int_0^t \int f\cdot u \, dxds , \tag{2}$$

for all $t > 0$.[1]

In [2], Caffarelli-Kohn-Nirenberg construct the so-called suitable weak solutions of (1) by the method of retarded mollification. They consider approximate solutions u_N, $N = 1,2, \ldots$, such that

$$u_N' + (\psi_\delta(u_N)\cdot\nabla)u_N - \Delta u_N + \nabla p_N = f \qquad \text{in } \Omega_T ,$$

$$\operatorname{div} u_n = 0 \qquad \text{in } \Omega_T ,$$

$$u_N|_{t=0} = u_0 \qquad \text{in } R^3 ,$$

where $\psi_\delta(u_N)$ is a certain retarded mollification of u_N (see [2]), $u_0 \in H$, $f \in L^2(0,T;H^{-1})$. They prove that there exists a subsequence u_N which converges in $L^2_{loc}(R^3 \times (0,T))$ to a weak solution of (1).

A simpler construction of a suitable weak solution is the one of Beirão da Veiga [1], made by considering the approximate problem

$$u_\varepsilon' + (u_\varepsilon\cdot\nabla)u_\varepsilon + \varepsilon\Delta^2 u_\varepsilon - \Delta u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon \qquad \text{in } \Omega_T ,$$

$$\operatorname{div} u_\varepsilon = 0 \qquad \text{in } \Omega_T ,$$

$$u_\varepsilon|_{t=0} = u_{0,\varepsilon} \qquad \text{in } R^3 ,$$

where $u_{0,\varepsilon} \in H^2 \cap V$, $u_{0,\varepsilon} \to u_0$ in L^2 as $\varepsilon \to 0$, $f_\varepsilon \in L^2(0,T;L^2)$ and $f_\varepsilon \to f$ in $L^2(0,T;H^{-1})$ as $\varepsilon \to 0$.

This paper deals with the asymptotic stability in L^2 of weak solutions of problem (1), with respect to perturbations of the initial data. Let $v_0 \in H$ and $f \in L^1(0,+\infty;H^{-1}) \cap L^2(0,+\infty;H^{-1})$. Assume that to v_0 and f it corresponds a sequence of approximate solutions, constructed as in [1] or in [2], weakly convergent in

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(1) For more informations about the weak solutions of the Navier-Stokes equations, see [3], [5], [6], [8].

$L^{2p/p-3}(0,+\infty;L^p)$, $p > 3$, to a weak solution v. Since $v \in L^{2p/p-3}(0,+\infty;L^p)$, it is the unique weak solution of (1) (see [8]).

We prove the following stability result:

Theorem A. Assume that the above conditions hold and let $u_0 \in H$. Then there exists a weak solution u of problem (1), corresponding to the initial velocity u_0 and the same external force f, such that

$$\|u(t) - v(t)\| \to 0 \qquad \text{as} \quad t \to \infty. \tag{3}$$

We prove theorem A for the weak solutions constructed in [1] and in [2]. Thus we show that such solutions, whatever be the initial velocity in H, converges, as t goes to infinity, to the given solution v. Our proof is inspired by the method of Schonbek [7] and Kajikiya-Miyakawa [4] for the study of the asymptotic behaviour of the weak solutions of (1).

In what follows C denotes various constants depending at most on p and the data u_0, v_0, f.

Proof of theorem A

The hypotheses on u_0 and f assure the existence of a global weak solution u (see [1], [2]). We start by showing a formal proof of property (3).

Let (u,p_1), (v,p_2) be two solutions corresponding to (u_0,f), (v_0,f) respectively. Then the difference $w = u - v$ satisfies the system

$$\begin{aligned} &w' + (u\cdot\nabla)w + (w\cdot\nabla)v - \Delta w + \nabla\pi = 0 && \text{in } \Omega_T, \\ &\operatorname{div} w = 0 && \text{in } \Omega_T, \\ &w|_{t=0} = w_0 && \text{in } R^3, \end{aligned} \tag{4}$$

where $\pi = p_1 - p_2$, $w_0 = u_0 - v_0$. Multiply equation (4) by w and integrate over R^3. Since u, v, w are divergence free, by integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 = \int (w\cdot\nabla)w\cdot v\ . \tag{5}$$

Consider the term on the right. By Hölder's inequality we get

$$(6)\qquad \left|\int (w\cdot\nabla)w\cdot v\right| \le \|w\|_r \|\nabla w\| \|v\|_p ,$$

where $\frac{1}{r} + \frac{1}{p} = \frac{1}{2}$. By interpolation we have

$$\|w\|_r \le C\, \|\nabla w\|^{3/p} \|w\|^{1-3/p} ,$$

hence, from (6), we get

$$\left| \int (w\cdot\nabla)w\cdot v\right| \le C\, \|\nabla w\|^{1+3/p} \|w\|^{1-3/p} \|v\|_p .$$

By introducing this last inequality in (5) and by using the Young's inequality, one obtains

$$(7)\qquad \frac{d}{dt} \|w\|^2 + \|\nabla w\|^2 \le C\, \|w\|^2 \|v\|_p^{\frac{2p}{p-3}} .$$

Consider now the Fourier transform $\hat{w}$ of w. Applying Plancherel's theorem to (7) gives

$$(8)\qquad \frac{d}{dt} \|\hat{w}\|^2 + \int |\xi|^2 |\hat{w}|^2 d\xi \le C\, \|\hat{w}\|^2 \|v\|_p^{\frac{2p}{p-3}} .$$

We now decompose the frequency domain into two time-dependent subdomains. Let $S(t)$ be the sphere in $\mathbb{R}^3$ centered at the origin with radius $r(t) = [\alpha/(t+1)]^{1/2}$, $\alpha > 1$. We have

$$\int |\xi|^2 |\hat{w}|^2 d\xi = \int_{S(t)} |\xi|^2 |\hat{w}|^2 d\xi + \int_{S(t)^c} |\xi|^2 |\hat{w}|^2 d\xi \ge$$

$$\frac{\alpha}{t+1} \int_{S(t)^c} |\hat{w}|^2 d\xi = \frac{\alpha}{t+1} \|\hat{w}\|^2 - \frac{\alpha}{t+1} \int_{S(t)} |\hat{w}|^2 d\xi .$$

Then from (8) we obtain

$$(9)\qquad \frac{d}{dt} \|\hat{w}\|^2 + \frac{\alpha}{t+1} \|\hat{w}\|^2 \le \frac{\alpha}{t+1} \int_{S(t)} |\hat{w}|^2 d\xi + C\, \|\hat{w}\|^2 \|v\|_p^{\frac{2p}{p-3}} .$$

Now, in order to estimate the first term on the right-hand side, we proceed as in [7] by considering the Fourier transform of equation (4):

$$(10)\qquad \hat{w}' + |\xi|^2 \hat{w} = G(\xi,t),$$

where $G(\xi,t) = -((u\cdot\nabla)w)^\wedge - ((w\cdot\nabla)v)^\wedge - i\xi\hat{\pi}$.

The solution of (10) is given by

$$\hat{w}(\xi,t) = e^{-|\xi|^2 t}\,\hat{w}_0(\xi) + \int_0^t e^{-|\xi|^2(t-s)} G(\xi,s)\,ds \ ,$$

so we have

$$(11) \qquad |\hat{w}(\xi,t)|^2 < 2e^{-2|\xi|^2 t}|\hat{w}_0(\xi)|^2 + 2t\int_0^t e^{-2|\xi|^2(t-s)}|G(\xi,s)|^2 ds \ .$$

We estimate the single terms of $G(\xi,t)$.

By integrating by parts and since u is divergence free, we have

$$|((u\cdot\nabla)\hat{w})\ | = \Big|\int_{R^3} \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(u_j w)e^{-i\xi\cdot x}dx\Big| =$$

$$\Big|\int_{R^3} \sum_{j=1}^{3} u_j w \xi_j e^{-i\xi\cdot x}dx\Big| < |\xi|\,\|u\|\|w\| \ .$$

In a similar way we obtain

$$|((w\cdot\nabla)v)^\wedge| < |\xi|\,\|v\|\|w\| \ .$$

In order to estimate the pressure term in $G(\xi,t)$, take the divergence of equation (4):

$$(12) \qquad -\Delta\pi = \sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}(u_i w_j + w_i v_j) \ .$$

Taking the Fourier transform of (12) gives

$$|\xi|^2\hat{\pi} = -\sum_{i,j=1}^{3} \xi_i \xi_j\,(u_i w_j + w_i v_j)^\wedge \ ,$$

from which one obtains

$$|\hat{\pi}| < (\|u\| + \|v\|)\|w\| \ .$$

Then we have

$$(13) \qquad \begin{aligned} |G(\xi,t)| &< 2|\xi|(\|u(t)\| + \|v(t)\|)\|w(t)\| \ , \\ &< 4|\xi|(\|u(t)\|^2 + \|v(t)\|^2) \ , \end{aligned}$$

where we have used

$\|w(t)\| \le \|u(t)\| + \|v(t)\|$. From the energy inequality (2) we have

$$\|u(t)\|^2 + \|v(t)\|^2 \le C \qquad \forall t > 0 ,$$

where the constant C is an increasing function of the L^2-norms of u_0, v_0 and of the $L^1(0,+\infty;H^{-1})$ and $L^2(0,+\infty;H^{-1})$-norms of f. Thus, from (13), we get

$$|G(\xi,t)| \le C|\xi| \qquad \forall t > 0 . \tag{14}$$

Introducing (14) in (11) and a direct calculation of the integral yield

$$|\hat{w}(\xi,t)|^2 \le 2\, e^{-2|\xi|^2 t}|\hat{w}_0(\xi)|^2 + Ct(1 - e^{-2|\xi|^2 t})$$

$$\le 2\, e^{-2|\xi|^2 t}|\hat{w}_0(\xi)|^2 + Ct .$$

Thus the first term in the right-hand side of (9) is estimated as follows:

$$\frac{\alpha}{t+1}\int_{S(t)}|\hat{w}|^2 d\xi \le \frac{2\alpha}{t+1}\int_{S(t)} e^{-2|\xi|^2 t}|\hat{w}_0|^2 d\xi + C\int_{S(t)} d\xi$$

$$\le \frac{2\alpha}{t+1}\int_{R^3} e^{-2|\xi|^2 t}|\hat{w}_0|^2 d\xi + C\,(t+1)^{-3/2} ,$$

and from (9) we obtain

$$\frac{d}{dt}\|\hat{w}\|^2 + \frac{\alpha}{t+1}\|\hat{w}\|^2 \le \frac{2\alpha}{t+1}\int_{R^3} e^{-2|\xi|^2 t}|\hat{w}_0|^2 d\xi + C(t+1)^{-3/2} + \tag{15}$$

$$+ C\|\hat{w}\|^2\|v\|_p^{\frac{2p}{p-3}} .$$

Multiplying by the integrating factor $(t+1)^{\alpha}$ gives

$$\frac{d}{dt}\left[(t+1)^{\alpha}\|\hat{w}\|^2\right] \le 2\alpha(t+1)^{\alpha-1}\int_{\mathbb{R}^3} e^{-2|\xi|^2 t}|\hat{w}_0|^2 d\xi$$

$$+ C(t+1)^{\alpha-3/2} + C_0(t+1)^{\alpha}\|\hat{w}\|^2\|v\|_p^{\frac{2p}{p-3}} .$$

Via the Gronwell inequality we thus obtain

$$(t+1)^{\alpha}\|\hat{w}\|^2 \le J(v)\|\hat{w}_0\|^2 + 2\alpha J(v)\int_0^t (\tau+1)^{\alpha-1}\int_{\mathbb{R}^3} e^{-2|\xi|^2\tau}|\hat{w}_0|^2 d\xi d\tau$$

$$+ CJ(v)(t+1)^{\alpha-1/2} ,$$

where $J(v) = \exp\left(C_0\int_0^{+\infty}\|v(s)\|_p^{\frac{2p}{p-3}} ds\right) < \infty.$

Finally we have

$$(16)\qquad \|\hat{w}\|^2 \le J(v)(t+1)^{-\alpha}\|\hat{w}_0\| + 2\alpha J(v)(t+1)^{-\alpha}\int_0^t(\tau+1)^{\alpha-1}\int_{\mathbb{R}^3} e^{-2|\xi|^2\tau}|\hat{w}_0|^2 d\xi d\tau +$$

$$CJ(v)(t+1)^{-1/2} .$$

The first and third term clearly converge to zero as $t \to \infty$. The same is easily proved for the second term by applying the ℓ'Hopital and Lebesgue theorems and using the fact that $\hat{w}_0 \in L^2$.

The same behavior can be proved for the weak solutions constructed in [1] and [2]. In fact, proceeding as above, one can prove inequality (16) for the approximate solutions of [1] and [2]. Then, by passing to the limit, one obtains (16) for such weak solutions, from which one has the behavior at $t \to \infty$ (for more detail see [4], [7]).

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper we study L^2 stability for weak solutions of the Navier-Stokes system of equations (1), in the whole space R^3. We assume that it is given a weak solution v satisfying certain hypotheses (essentially $v \in L^{2p/p-3}(0; +\infty; L^p)$, $p > 3$) and we prove that, to an arbitrarily large perturbation of his initial velocity, it corresponds a weak solution u such that the L^2-norm $\|u(t) - v(t)\|$ converges to zero as time t tends to infinity.

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